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Magnetic-field-limited currents

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An upper limit on the net current of a charged particle beam is derived by requiring the energy per unit length in the magnetic field to be less than that in the particles. The limit is calculated for five different current profiles. It is shown that an arbitrarily large net current can propagate in a large diameter ring. The results are found to be closely related to the Alfvén limit. A limit on the forward current in a conductor is calculated, which defines a magnetic inhibition time.

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In this paper, I return to the problem considered by Alfvén [1] in 1938, the limitation of current flow by magnetic-field generation. Alfvén considered an arbitrarily large, cylindrical beam of charged particles with a uniform current density and no net charge density. (Cylindrical beams that vary only in radius r will be assumed throughout, although the following argument could be applied to a beam with almost any cross section.) The magnetic field in this beam would increase linearly with radius. Alfvén calculated the trajectories of the particles in this magnetic field (given by elliptic integrals), shown in Fig. 1, and found that beyond a certain radius the particles moved backwards. This means that the forward propagating beam initially envisaged must have a finite radius, corresponding to a maximum current

$$I_{A\text{-}U}=1.65\frac{4\pi}{q\mu_0}p, \tag{1}$$

where q is the charge of the particles, p is the momentum of the particles, and the subscript U indicates that the limit refers to a uniform current density. Equation (1) differs from the expression given by Alfvén [Eq. (4)] as it is in SI units, it does not assume that the particle's kinetic energy is much greater than its rest mass energy (ultrarelativistic case), and it has been expressed in terms of momentum instead of energy. As pointed out by Alfvén, this is not a limit on the forward current that can propagate, but a limit on the net current for a given current profile, which has not been determined self-consistently. For example, as can be seen in Fig. 1, the magnetic field leads to a return current that will allow a higher forward current to propagate, and that will modify the current profile, modifying the limit. A possibility considered by Alfvén was that the particles returning outside what he called the direct beam would allow particles at a larger radius to propagate, giving a higher forward current. In essence this argument states that a higher current can be carried by using separate beams, to which separate limits can be applied. In this case, the beams are concentric, hollow cylinders, which is the form filamentation takes in rotational symmetry. The lack of self-consistency should not be seen as a problem, as the aim of the analysis is not a self-consistent treatment of beam propagation, but to address the possibility, or rather impossibility, of propagating a given beam. If a beam cannot propagate then clearly it must be significantly modified.

Before moving on from Alfvén's work, it should be mentioned that Eq. (1) is not the result that is normally quoted nor is it Alfvén's derivation that is normally given. Normally a finite beam is considered, and the current at which the Larmor radius of a particle at the edge of the beam would be less than half the beam radius is calculated. This gives the Alfvén-Lawson limit (with complete charge neutralization and no magnetic neutralization) [2]

$$I_{A\text{-}1/r}=\frac{4\pi}{q\mu_0}p. \tag{2}$$

Alfvén gave this result as an order of magnitude estimate, dropping the factor of 1.65 [Eq. (5)]. In this derivation the dependence on current profile is not clear. In fact, by assuming that the Larmor radius calculated at the edge of the beam will determine when a particle is turned back, it has been assumed that the magnetic field is uniform, which requires a current density inversely proportional to radius, hence the $1/r$

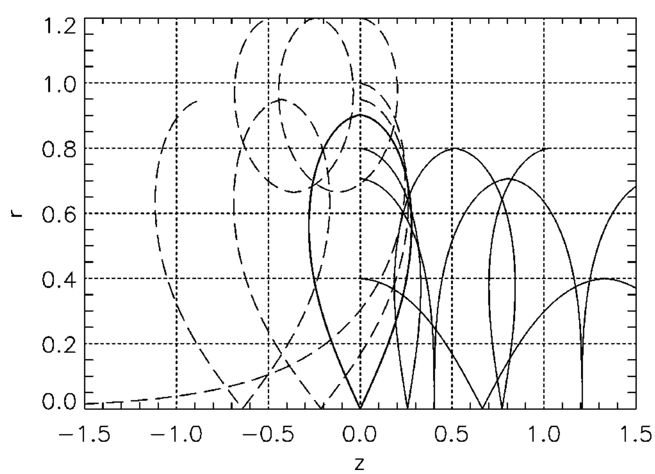

FIG. 1. Particle trajectories in the magnetic field of a beam with a uniform current density. Particles start at $z=0$ moving parallel to the axis with momentum p. Distances are normalized so that the total current within a radius r is $(8\pi/q\mu_0)pr^2$. A particle starting at $r\approx0.9$ (thick line) has no net motion in the axial direction and defines the Alfvén limit. Particles starting at larger radii (dashed lines) have a net backward motion and those at smaller radii (solid lines) a net forward motion.

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subscript. For a uniform current density, this is the current at which no particle moves backwards. If the beam is emitted from a source that absorbs or reflects the particles, this would be the limit.

The Alfvén limit has also been calculated for the Bennet profile by Honda [3]. Bennet [4] considered a cylindrically symmetric electron beam, neutralized by a counterstreaming ion beam, with a constant axial velocity, a constant transverse temperature, and no axial temperature. He found equilibrium solutions in which the radial force from the magnetic field, generated by the axial current, balanced that from the transverse pressure. Honda considered an approximate, particular solution for the current density given by Bennet that has the form $j_0/(1+r^2/R^2)^2$. This case differs from those considered previously, in that the beam has an infinite extent, but a finite total current ($\pi R^2 j_0$). The limit thus depends on radius, going to 0 at ∞. At R, which contains half the total current, Honda gives, from a numerical solution,

$$I_{A\text{-}B}=1.27\frac{4\pi}{q\mu_0}p. \tag{3}$$

However, the equilibrium current is fixed by the beam parameters, so this treatment just gives the maximum radius of the equilibrium. For zero ion temperature and propagation velocity this current is $(16\pi/q\mu_0)(kT/v)$, where kT is the transverse electron temperature. It will only exceed Eq. (3) when the propagation velocity is comparable to the transverse thermal velocity, which violates one of Bennet's initial assumptions. The maximum radius of the equilibrium is thus much greater than R, so the current will not be significantly reduced. Equation (3) should instead be considered for such a current profile that is not in equilibrium. What this result does show, along with that of Eq. (2), is that the pinching caused by the magnetic field will lower the current limit. As a result, the limit for a uniform beam [Eq. (1)] is unlikely to be reached.

Let us now return to the case considered by Alfvén, and take a different approach, considering the radius at which the energy per unit length in the magnetic field equals that of the particles. At this point all of the energy of the particles should have gone into the magnetic field and there would be no current. The energy per unit length in the magnetic field, for a uniform current density, is $\mu_0 I^2/16\pi$. The kinetic energy per unit length of the particles that are generating this magnetic field is $\langle K\rangle I/qv$, where $\langle K\rangle$ is their mean kinetic energy and v is the beam propagation velocity (if this varies with radius, an average given by the current divided by the charge per unit length should be used). Equating these gives

$$I_U=\frac{16\pi}{q\mu_0}\frac{\langle K\rangle}{v}. \tag{4}$$

This result can be readily extended to any integrable current profile. Writing the radial dependence of the total current within a radius r, $I(r)$, as

$$f(\rho=r/R)\equiv\frac{I(r)}{I(R)}, \tag{5}$$

the limit at radius R can be written in the simple form

$$I=\frac{1}{\int_0^1 d\rho f^2(\rho)/\rho}\frac{4\pi}{q\mu_0}\frac{\langle K\rangle}{v}. \tag{6}$$

Evaluating Eq. (6) for $j\propto 1/r$ and the Bennet profile (at R) is trivial, giving

$$I_{1/r}=\frac{8\pi}{q\mu_0}\frac{\langle K\rangle}{v} \tag{7}$$

and

$$I_B=2.59\frac{4\pi}{q\mu_0}\frac{\langle K\rangle}{v}. \tag{8}$$

The close relationship between this limit and the Alfvén limit is immediately obvious. This is perhaps not surprising, as they are just different approaches to the same problem, one based on particle trajectories and the other on the conservation of energy.

To look more closely at the relationship between the two limits, consider a monoenergetic beam where all of the particles are moving in the same direction, which was the point of departure for Alfvén. We then have $\langle K\rangle/v=p/(1+1/\gamma)$, where γ is the Lorentz factor of the particles. The value of $\langle K\rangle/v$ thus varies from $p/2$ in the nonrelativistic limit to p in the ultrarelativistic limit. The ratio of this limit to the Alfvén limit is then 1.21–2.42 for a uniform current density, 1–2 for $j\propto 1/r$, and 1.02–2.04 for the Bennet profile. The dependence of this limit on the current profile, and its order of magnitude in this case, is practically the same as that of the Alfvén limit. It might be expected that a more accurate calculation of the current limit would give a result a factor of 2 lower, as this is the point at which the initial energy should be equally divided between the energy remaining in the particles and the magnetic field. The fact that the two limits can be the same is perhaps surprising. The ratio between the two limits will be higher if the forward velocity of the particles is lower than their total velocity, or in other words, if the beam has a temperature. For the ratio to be much greater than 1 the propagation velocity would have to be much lower than the thermal velocity. This is the case normally considered for current flow in a conductor, but in such cases effects other than the magnetic field usually dominate.

Apart from the simplicity of its calculation, and the fact that it is clearly an absolute upper limit on the net current, this limit also has the advantage over the Alfvén limit of being directly applicable to beams that are not monoenergetic. The main limitation that remains is the dependence on current profile, which will obviously be changed by the magnetic field. This raises the question is there a current profile that significantly increases this limit (other than a series of separate beams)? The answer is clearly yes. The magnetic-field generated by a current element falls as $1/r$, so the magnetic-field energy in that element ($\propto B^2 r$) will also fall as $1/r$, therefore by concentrating the current at large radii the current limit will be lowered. We have already seen that current densities that peak on axis give a lower limit than a uniform current density. To have a look at the other extreme, consider a uniform ring of current with radius r and thickness R, such that $r \gg R$. The current limit in this case is

$$I_R \approx \frac{r}{R} \frac{12\pi}{q\mu_0} \frac{\langle K \rangle}{v}, \quad r \gg R, \tag{9}$$

to first order in R/r. The Alfvén limit in this case can be obtained using the results for a uniform current density (Fig. 1), the limit being defined by the particle that reaches inside of the ring with no axial velocity, giving

$$I_{A-R} \approx \frac{r}{R} \frac{4\pi}{q\mu_0} p, \quad r \gg R. \tag{10}$$

This yet again demonstrates the close relationship between the two limits. It should be mentioned that Alfvén also calculated a limit for the current passing a circular surface surrounding the direct beam. He assumed that the magnetic field at the edge of the region could not exceed that at the edge of the beam. Although this limit [Eq. (8)] differs from Eqs. (9) and (10), it is also proportional to the radius of the surface. A less extreme example is the Hammer-Rostoker [5] equilibrium. They obtained a finite radius, equilibrium solution for a monoenergetic beam, where all particles have the same axial momentum. The current density has the form $j_0 \mathcal{I}_0(r/R)$, where $\mathcal{I}_0$ is the zeroth order modified Bessel function, which is 1 on axis and increases exponentially at large radii. This gives a limit

$$I_{HR} = \frac{1}{1 - \mathcal{I}_1'^2/\mathcal{I}_1^2 + R^2/r^2} \frac{8\pi}{q\mu_0} \frac{\langle K \rangle}{v}, \tag{11}$$

where the prime denotes the derivative and the argument of the modified Bessel functions (r/R) has been suppressed. The first factor on the right hand side of Eq. (11) tends to r/R as r tends to infinity, so also gives a limit proportional to the radius of the beam. The equilibrium current is $(r/R)(\mathcal{I}_1)(2\pi/q\mu_0)p$, with R given by the collisionless skin depth on axis. This increases exponentially as r tends to infinity, so there is a limit on the equilibrium current. Obtaining the Alfvén limit for this profile requires a numerical solution. These results show that it is possible to have a current profile that allows an arbitrarily large current to propagate in a single beam. They also show that beam hollowing and the separation of excess current into rings, as Alfvén considered, are very efficient means of allowing a beam with excess current to propagate.

A current limit due to electric field generation in a beam with a net charge density could also be calculated. As an example, consider a beam with no charge neutralization and a uniform propagation velocity. Considering just the radial electric field, which is the only component in an infinitely long, rotationally symmetric beam, gives the same result as for the magnetic field, but multiplied by v^2/c^2. Including both the electric and magnetic fields multiplies the magnetic field limit by $1/(1+c^2/v^2)$. The electric field lowers the limit and imposes a greater limitation than the magnetic field. This is on the contrary to the result given by Lawson [3,5], where the electric field increases the limit, eliminating it for no, or sufficiently low, charge neutralization. This is because the radial electric field opposes the inwards force from the magnetic field. With no charge neutralization we have $E/B > c$, so the particles are not turned back. This result can be obtained if we assume that the energy of the particles is given by their kinetic energy plus their potential energy. However, the potential energy will only become available to the magnetic field if the particles separate completely, destroying the beam. If we assume that there is initially no electric field, as we did for the magnetic field, then the electric field energy must come from the particles. As the source of the electric field is charge not current, it is best considered in terms other than a current limit.

There is, however, an apparent problem with this limit for a finite radius beam. Only the magnetic-field energy inside the beam has been considered, but outside the beam there is a magnetic field given by $\mu_0 I/2\pi r$. If the energy in this field is included, then the limit will fall with radius, going to 0 at ∞. This field is normally considered to increase the Alfvén limit [1,2]. The same problem was encountered in calculating both limits for the Bennet profile. The reason for this is that in order to maintain the propagation of a beam there must be a return current. What this limit gives us is the current at which this return current must flow within the beam itself. The interaction between the beam and the return current cannot then be ignored, for example, by assuming that the beam is flowing inside a conducting pipe. In the absence of a return current a beam would propagate for a limited time, which will fall as current increases. A time dependent current limit due to magnetic-field generation can be derived by using retarded currents. For example, the limits calculated here correspond to roughly a light transit time across the beam radius. However, this is beyond the scope of this paper.

Finally, I will consider the propagation of a charged particle beam in a conductor. This could be a plasma, a metal, or any material if the current is high enough for breakdown to occur. In this case forward currents far higher than the above limits can propagate, due to the presence of a return current from the conductor. The simplest model for this current density $\mathbf{j}_c$ is to use the basic Ohm's law for the electric field, $\mathbf{E} = \eta \mathbf{j}_c$, where η is resistivity. This assumes that the dynamics of the conduction electrons is dominated by collisions, so considerations of magnetic-field limitation of the return current do not apply. Now if the beam current is much higher than the limiting value it must be almost exactly balanced by the background current, so we can use $\mathbf{j}_c \approx -\mathbf{j}$. Substituting the resulting expression for the electric field into Faraday's law gives

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \eta \mathbf{j}, \tag{12}$$

from which the magnetic field, and hence the forward current limit, can be obtained. Assuming a constant resistivity and a fixed current density the calculation is straightforward. For

Eq. (12) to be valid the current density must have a smooth profile, so it can only be applied to the example of the Bennet profile. Assuming only an axial current density gives an azimuthal magnetic field

$$B=\frac{4r/R^2}{(1+r^2/R^2)^3}\,\eta j_0 t. \tag{13}$$

The forward current limit is then

$$I_{C\text{-}B}=\left(\frac{t_D}{t}\right)^2\frac{2.5\pi}{q\mu_0}\frac{\langle K\rangle}{v}, \tag{14}$$

where C indicates that the limit applies in a conductor and t_D is the magnetic diffusion time [3]

$$t_D=\frac{\mu_0 R^2}{\eta}. \tag{15}$$

It gives the time scale for which the beam and conduction current densities remain approximately in balance ($t \ll t_D$ has been implicitly assumed). As this limit is proportional to $1/t^2$, it gives the maximum time for which a forward current greater than that given by Eq. (8) can propagate. For a current that exceeds Eq. (8) by a factor f, this inhibition time is

$$t_I=\frac{t_D}{1.43\sqrt{f}}. \tag{16}$$

Since the magnetic field significantly modifies the particle trajectories before this limit is reached and reduces the beam's radius, and hence the magnetic diffusion time, this must be considered as an upper limit on the inhibition time. This indicates that even a current equal to that given by Eq. (8) can only be maintained for a fraction of a magnetic diffusion time. Since the magnetic diffusion time increases as the square of the beam radius, a desired propagation time can be achieved by making the beam wide enough. Evaluating the Alfvén limit in this case requires numerical techniques.

In conclusion, a current limit due to the generation of magnetic field has been derived by requiring the energy per unit length in the magnetic field to be less than that in the current carrying particles. It has been evaluated for five different current profiles. In particular, it was shown that an arbitrarily large current can propagate in a large diameter ring. The limit has been shown to closely follow the Alfvén limit. However, it has the advantages that it is much easier to calculate, is directly applicable to beams that are not monoenergetic, and, being based on energy conservation, is unambiguously an upper limit on the net current. A forward current limit for propagation in a conductor has been calculated, which defines a magnetic inhibition time.

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